Mathematics exercises for exam preparation:

A. Integrals; B. Calculation of Area

A. Evaluate the following integrals.

1.

$$\int e^{\frac{x}{2}}\,dx = \int e^{\frac{x}{2}}\ 2\,d\frac{x}{2} = 2\int e^{\frac{x}{2}}\ d\frac{x}{2} = 2e^{\frac{x}{2}} + C$$

2.

$$\int 5^x\,dx = \int e^{\ln 5^x}\,dx = \int e^{\mathrm{x}\ln 5}\,dx = \frac{1}{\ln 5}\int e^{\mathrm{x}\ln 5}\,dx\ln 5 = \frac{e^{x\,ln\,5}}{\ln 5} + C$$

3.

$$\int e^{-3}\,dx = \frac{1}{e^3}\int 1\,dx = \frac{x}{e^3} + C$$

4.

$$\int e^{x^3}x^2\,dx = \int e^{x^3}\frac{dx^3}{3} = \frac{1}{3}\int e^{x^3}dx^3 = \begin{pmatrix} u = x^3 \\ du = dx^3 \end{pmatrix} = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C$$

$$= \frac{1}{3}e^{x^3} + C$$

5.

$$\int \frac{\sin x}{(\cos x)^3}dx = \begin{pmatrix} -\sin x\,dx = du \\ u = \cos x \end{pmatrix} = \int \frac{-du}{u^3} = -\frac{u^{-3+1}}{-3+1} + C = \frac{1}{2(\cos x)^2} + C$$

6.

$$\int \sqrt[3]{5-6x}\ dx = \begin{pmatrix} u = 5-6x \\ du = -6dx \end{pmatrix} = \int \sqrt[3]{u}\frac{du}{-6} = -\frac{1}{6}\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = -\frac{1}{8}u^{\frac{4}{3}} + C$$

$$= -\frac{1}{8}(5-6x)^{\frac{4}{3}} + C$$

7.

$$\int \cos\frac{x}{4}dx = \int \cos\frac{x}{4}d\frac{x}{4}\ 4 = \begin{pmatrix} u = x/4 \\ du = dx/4 \end{pmatrix} = 4\int \cos u\,du = 4\sin u + C = 4sin\frac{x}{4} + C$$

8.

$$\int \sin x\,(\cos x)^3\,dx = \begin{pmatrix} \cos x = u \\ -\sin x\,dx = du \end{pmatrix} = \int -u^3\,du = -\frac{u^{3+1}}{3+1} + C = -\frac{(\cos x)^4}{4} + C$$

9.

$$\int \frac{dx}{x\,(1+\ln x)}dx = \begin{pmatrix} u = \ln x \\ du = dx\ \frac{1}{x} \end{pmatrix}$$

$$= \int \frac{du}{1+u} = \int \frac{d(1+u)}{1+u} = \ln|1+u| + C = \ln|1+\ln x| + C$$

10.

$$\int \frac{dx}{(\arcsin x)^2\sqrt{1-x^2}} = \begin{pmatrix} u = \arcsin x \\ du = dx\ \frac{1}{\sqrt{1-x^2}} \end{pmatrix} = \int \frac{du}{(u)^2} = \frac{u^{-2+1}}{-2+1} + \mathrm{C} = \frac{-1}{u} + \mathrm{C} =$$

$$= -\frac{1}{\arcsin x} + C$$

11.

$$\int x \sin x \; dx = \begin{pmatrix} \sin x = v' \\ -\cos x = v \\ x = u \\ dx = du \end{pmatrix} = x(-\cos x) - \int 1(-\cos x)\, dx = -x\cos x + \sin x + C$$

12.

$$\int x^2 \cos x \; dx = \begin{pmatrix} \cos x = v' \\ \sin x = v \\ x^2 = u \\ 2xdx = du \end{pmatrix} = x^2 \sin x - \int 2\boldsymbol{x \sin x\, dx}$$

$= |integral\ in$ **bold is identical to previous integral** $|$

$$= x^2 \sin x - 2(-x\cos x + \sin x) + C$$

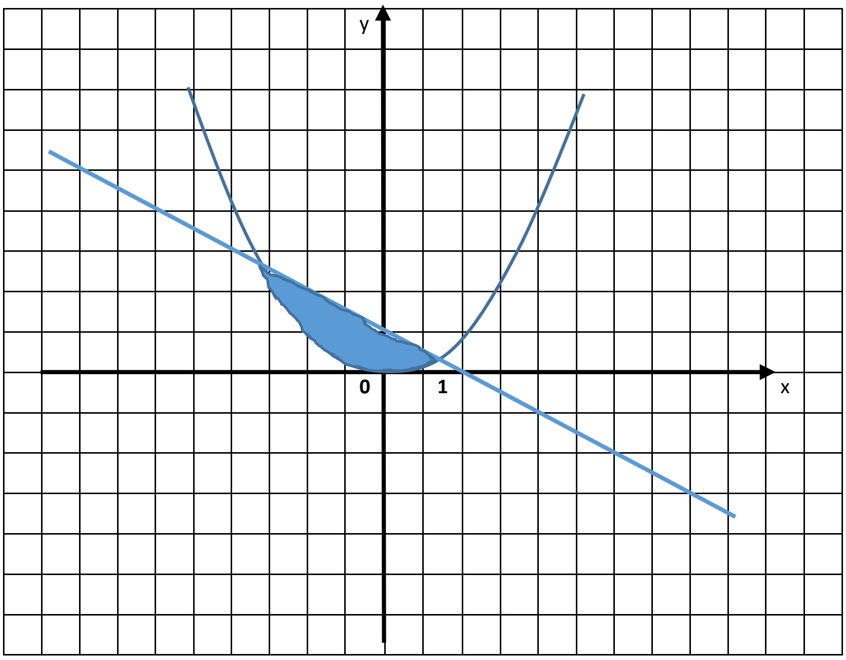
13.

$$\int x e^x \, dx = \begin{pmatrix} x = u \\ dx = du \\ e^x = v \\ e^x = dv \end{pmatrix} = xe^x - \int e^x \, dx = xe^x - e^x + C = e^x\,(x-1) + C$$

B. Calculation of Area.

1.

a) Sketch the area of the region enclosed by the curves of $f(x) = \frac{x^2}{2}$ and $g(x) = \frac{1}{2} - \frac{x}{2}$.



b) Find the area of the region enclosed by the curves of $f(x) = \frac{x^2}{2}$ and $g(x) = \frac{1}{2} - \frac{x}{2}$.

$$\frac{1}{2} - \frac{x}{2} = \frac{x^2}{2}$$

Intersection points: $x1 = \frac{-1-\sqrt{5}}{2}$; $x2 = \frac{-1+\sqrt{5}}{2}$

$$Area = \int_{x1}^{x2} \frac{1}{2} - \frac{x}{2} - \frac{x^2}{2}\, dx = \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{6}\ \Big|_{x1}^{x2}$$

$$= \frac{-1+\sqrt{5}}{4} - \frac{-1-\sqrt{5}}{4} - \frac{(-1+\sqrt{5})^2}{16} + \frac{(-1-\sqrt{5})^2}{16} - \frac{(-1+\sqrt{5})^3}{48} + \frac{(-1-\sqrt{5})^3}{48} = 0{,}93$$

3. Draw the function $f(x) = e^{2t+1}$ on the interval [-1;1] and evaluate the integral.

$$\int_{-1}^{1} e^{2t+1}dt = \int_{-1}^{1} \frac{1}{2} e^{2t+1} d(2t+1) = \frac{1}{2} e^{2t+1}\Big|_{-1}^{1} = \frac{e^3}{2} - \frac{1}{2e^1} \approx 9{,}86$$